Two kinds of nonlinear inversions for Bell polynomials

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The present talk gives two kinds of new nonlinear inverse relations of the ordinary Bell polynomials. Both are derived from the classical Lagrange expansion theorem. The first one is as follows.

Theorem 1 For any integers $n \geq 1$ and $a, b \in \mathbb{C}$, and any sequence $\{x_n\}_{n\geq 1}$, define

$$y_n(b) = \frac{b}{an+b} \sum_{k=1}^{n} \binom{-an-b}{k} B_{n,k}(x_1, x_2, \cdots, x_{n-k+1}). \tag{1}$$

Then

$$x_n = \sum_{k=1}^{n} \frac{(-1)^k}{k!b^k} \left\{ \prod_{i=1}^{k-1} (an+1+bi) \right\} B_{n,k}(y_1(b), y_2(b), \cdots, y_{n-k+1}(b)). \tag{2}$$

Vice versa.

As applications, it not only provides a simple proof but also a new general version for Birmajer–Gil–Weiner's Bell polynomial inverse relations published by Electronic J. of Combin. 19(4) (2012) #P34, which is new and somewhat unusual inverse relation .

Furthermore, in a similar manner, we find another kind of nonlinear inverse relations which is infinite and arises from Terence Tao's power series problem (see https://terrytao.wordpress.com/2016/10/23/another-problem-about-power-series/).

Theorem 2 For any sequences $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$, the nonlinear system of relations

$$y_n = \sum_{k=n}^{\infty} \frac{1}{k+1} \binom{k+1}{n} B_{k,k-n+1}(x_1, x_2, \cdots, x_n) \tag{3}$$

is equivalent to the nonlinear system

$$x_n = \sum_{k=n}^{\infty} \frac{(-1)^{k-n}}{k+1} \binom{k+1}{n} B_{k,k-n+1}(y_1, y_2, \cdots, y_n). \tag{4}$$